Topics in Algebra - Singularities in Positive Characteristic

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Semester 2

1.1 Review of Algebraic Geometry: Schemes

Remark 1.1.1. The model for us is schemes. Recall that, roughly, schemes are topological spaces with sheaves. A presheaf $\mathcal{F}$ of abelian groups (eventually, rings) on a topological space X is an assignment of an abelian group (ring) $\mathcal{F}(U)$ for each open set $U \subseteq X$ and restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ for $V \subseteq U$. We denote sections by $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$ and the image of $s \in \Gamma(U, \mathcal{F})$ in $\Gamma(V, \mathcal{F})$ as $s|_V$.

Example 1.1.2. If X is a complex manifold, then $\mathcal{O}_X$ is the presheaf of holomorphic functions, and

$$\mathcal{O}_X(U) = \{f : U \to \mathbf{C} \text{ holomorphic}\}.$$

Example 1.1.3. Fix an abelian group A. We can form a presheaf $\mathcal{F}_A(U) = A$ with all restriction maps $\mathcal{F}_A(U) \xrightarrow{\text{id}} \mathcal{F}_A(V)$.

Example 1.1.4. Fix $x \in X$ and an abelian group A. We get a presheaf

$$\kappa_A(x)(U) = \begin{cases} A & \text{if } x \in U; \\ 0 & \text{if } x \notin U. \end{cases}$$

The presheaf $\kappa_A(x)$ is called the skyscraper presheaf at x.

Definition 1.1.5 (sheaf)**.** A presheaf $\mathcal{F}$ is a **sheaf** provided for each open cover $\{U_i\}$ of X and $s_i \in \Gamma(U_i, X)$ with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, there is a unique $s \in \Gamma(X, \mathcal{F})$ such that $s|_{U_i} = s_i$.

Example 1.1.6. Of the previous three examples **Example 1.1.2**, **1.1.3**, and **1.1.4**, the presheaf of holomorphic functions is a sheaf and the skyscraper presheaf is a sheaf, but the constant presheaf is not a sheaf. Let $A = \mathbf{Z}$; consider a connected topological space X and disjoint open subsets U_1 and U_2 in a fixed cover $\{U_i\}$ of X such that $s_1 = 2$ on U_1 and $s_2 = 3$ on U_2. Then $s_1|_{U_1 \cap U_2} = s_2|_{U_1 \cap U_2}$ since $U_1 \cap U_2 = \emptyset$, but there is no global section s for which $s|_{U_1} = s_1$ and $s|_{U_2} = s_2$.

Definition 1.1.7 (sheafification)**.** To any presheaf $\mathcal{F}$, there is a unique sheaf $\mathcal{F}^+$, the **sheafification** of $\mathcal{F}$, with the universal property that there is a map of presheaves $\mathcal{F} \to \mathcal{F}^+$ so that for any sheaf $\mathcal{G}$ and map $\mathcal{F} \to \mathcal{G}$, there is a diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}^+ \\ & \searrow & \downarrow \\ & & \mathcal{G} \end{array}$$

Remark 1.1.8. Sheafification is used to define an abelian category of sheaves. For example, for a map of sheaves $\mathcal{F} \to \mathcal{G}$, we define $\operatorname{im}(\mathcal{F} \to \mathcal{G})$ to be the sheafification of the presheaf $U \mapsto \operatorname{im}(\mathcal{F}(U) \to \mathcal{G}(U))$.

☡ **Warning! 1.1.9.** A sequence of presheaves

$$\cdots \to \mathcal{F}^{i-1} \to \mathcal{F}^i \to \mathcal{F}^{i+1} \to \cdots$$

is exact if and only if

$$\cdots \to \mathcal{F}^{i-1}(U) \to \mathcal{F}^i(U) \to \mathcal{F}^{i+1}(U) \to \cdots$$

is exact for all open sets $U \subseteq X$. But this does *not* characterize exactness as sheaves. Instead, one uses stalks, which are, for $x \in X$, the group

$$\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U).$$

Thus

$$\cdots \to \mathcal{F}^{i-1} \to \mathcal{F}^i \to \mathcal{F}^{i+1} \to \cdots$$

is exact as sheaves if and only if

$$\cdots \to \mathcal{F}^{i-1}{}_x \to \mathcal{F}^{i}{}_x \to \mathcal{F}^{i+1}{}_x \to \cdots$$

is exact.

Definition 1.1.10 (ringed space)**.** A **ringed space** is a pair $(|X|, \mathcal{O}_X)$ of a topological space and a sheaf of rings $\mathcal{O}_X$, called the structure sheaf.

Example 1.1.11. Let $|X|$ be any complex manifold and set $X = (|X|, \mathcal{O}_X)$ where $\mathcal{O}_X$ is the sheaf of holomorphic functions.

Example 1.1.12. Fix any commutative ring R. We get a topological space

$$\operatorname{Spec} R = \{\mathfrak{p} \subseteq R \mid \mathfrak{p} \text{ is a prime ideal}\}.$$

For $f \in R$, set $V(f) = \{\mathfrak{p} \in \operatorname{Spec} R \mid f \in \mathfrak{p}\} \subseteq \operatorname{Spec} R$. Declare $U_f = \operatorname{Spec} R \setminus V(f)$ to be a basis of open sets. We can also define a sheaf $\mathcal{O}_X$ by declaring $\mathcal{O}_X(U_f) = R_f = R[1/f]$. Explicitly, for $U \subseteq \operatorname{Spec} R$,

$$\Gamma(U, \mathcal{O}_X) = \left\{ s : U \to \bigcup_{\mathfrak{p} \in U} R_{\mathfrak{p}} \mid s(\mathfrak{p}) \in R_{\mathfrak{p}} \text{ and locally } s = \frac{f}{g} \text{ for } f, g \in R \right\}.$$

We get a ringed space $X = \operatorname{Spec} R = (|\operatorname{Spec} R|, \mathcal{O}_X)$.

Definition 1.1.13 (morphism of ringed spaces)**.** Given two ringed space $X = (|X|, \mathcal{O}_X)$ and $Y = (|Y|, \mathcal{O}_Y)$, a **map** $X \to Y$ is a continuous function $\varphi : |X| \to |Y|$ and a map of sheaves $\varphi^\sharp : \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$, where $\varphi_* \mathcal{O}_X$ is a sheaf on $|Y|$ defined by, for $U \subseteq |Y|$, $\varphi_* \mathcal{O}_X(U) = \mathcal{O}_X(\varphi^{-1} U)$. Thus a map is $(\varphi, \varphi^\sharp) : X \to Y$, which we denote $\varphi : X \to Y$ (deal with it).

Remark 1.1.14. We get a natural notion of isomorphism for ringed spaces; $X \cong Y$ if and only if there are maps $\varphi : X \to Y$ and $\psi : Y \to X$ with $\varphi \circ \psi$ and $\psi \circ \varphi$ identities.

Definition 1.1.15 (affine scheme)**.** A ringed space $X = (|X|, \mathcal{O}_X)$ is an **affine scheme** if $X \cong \operatorname{Spec} R$ for some ring R.

Definition 1.1.16 (scheme)**.** We say X is a **scheme** if there is an open cover $\{U_i\}$ of X with

$$\left(U_i, \mathcal{O}_X|_{U_i}\right) \cong \operatorname{Spec} R_i$$

for commutative rings R_i.

Remark 1.1.17. For $X = \operatorname{Spec} R$, we view R as functions on X. For example, if $R = \mathbf{Z}$, then $\operatorname{Spec} \mathbf{Z} = \{(0), (p) \mid p \in \mathbf{Z} \text{ is prime}\}$. So, in this perspective, $f = 12$ is a function. What is $f((5))$? It is $f((5)) = 12 \bmod 5 \equiv 2 \bmod 5 \in \mathbf{Z}/5\mathbf{Z}$.

Also note that not all points of $\operatorname{Spec} \mathbf{Z}$ are closed. In fact, $\overline{(0)} = \operatorname{Spec} \mathbf{Z}$. We call (0) a generic point.

Example 1.1.18. Here is an example of a non-affine scheme. Fix $S = \mathbf{C}[x_0, ..., x_n]$. Note that S is a graded ring;

$$S = \bigoplus_{d \geq 0} [S]_d$$

where each $[S]_d$ is the $\mathbf{C}$-vector space generated by $\underline{x}^{\underline{\lambda}} = x_0{}^{\lambda_0} \cdot x_1{}^{\lambda_1} \cdots x_n{}^{\lambda_n}$ with $\lambda_0 + \lambda_1 + \cdots + \lambda_n = d$.

For instance, if $n = 2$, then $S = \mathbf{C}[x, y]$, so then

$$\begin{aligned}
[S]_0 &\cong \mathbf{C} = \mathbf{C} \cdot 1 \\
[S]_1 &\cong \mathbf{C}^2 = \mathbf{C} \cdot x \oplus \mathbf{C} \cdot y \\
[S]_2 &\cong \mathbf{C}^3 = \mathbf{C} \cdot x^2 \oplus \mathbf{C} \cdot xy \oplus \mathbf{C} \cdot y^2;
\end{aligned}$$

that is, $[S]_d$ are monomials of total degree d, and $\dim_{\mathbf{C}}[S]_d = \binom{n+d}{d}$. We call $f \in S$ homogeneous if $f \in [S]_d$ for some d. A prime $\mathfrak{p} \subseteq S$ is homogeneous if it can be generated by homogeneous elements.

Now, define $\operatorname{Proj} S = \{\mathfrak{p} \subseteq S \mid \mathfrak{p} \text{ is a homogeneous prime}\}$. Any homogeneous $f \in S$ defines $V(f) = \{\mathfrak{p} \in \operatorname{Proj} S \mid f \in \mathfrak{p}\}$, and we let $U_f = \operatorname{Proj} S \setminus V(f)$ give a basis for the Zariski topology on $\operatorname{Proj} S$. We get a sheaf $\mathcal{O}_X$ with

$$\Gamma(U, \mathcal{O}_X) = \left\{ s : U \to \bigcup_{\mathfrak{p} \in U} S_{\mathfrak{p}} \mid s(\mathfrak{p}) \in S_{\mathfrak{p}} \text{ and locally } s = \frac{f}{g} \text{ with } \deg f = \deg g \right\}.$$

Note that $\Gamma(X, \mathcal{O}_X) \cong \mathbf{C} \ncong S$; we have fewer global sections than what is necessary for X to be affine.

Note also that each x_i in $S = \mathbf{C}[x_0, ..., x_n]$ defines

$$U_i = U_{x_i} = \operatorname{Proj} S \setminus V(x_i) \cong \operatorname{Spec} \mathbf{C}\left[\frac{x_0}{x_i}, \frac{x_1}{x_i}, ..., \widehat{\frac{x_i}{x_i}}, ..., \frac{x_n}{x_i}\right] \cong \mathbf{A}^n_{\mathbf{C}}.$$

The set $\{U_i\}$ form a cover of $\operatorname{Proj} S$. This identifies $\operatorname{Proj} S$ with $\mathbf{P}^n_{\mathbf{C}}$, projective space. We call $\{U_i\}$ the standard affine charts of projective space.

Definition 1.1.19 (irrelevant ideal)**.** Let $S = \mathbf{C}[x_0, ..., x_n]$. The ideal $\mathfrak{m} = (x_0, ..., x_n)$ is called the **irrelevant ideal** because $\mathfrak{m} \notin \operatorname{Proj} S$.

Example 1.1.20. Set $X = \mathbf{P}^2_{\mathbf{C}} \cong \operatorname{Proj} \mathbf{C}[x, y, z]$. This has covers

$$\begin{aligned} U_x &= \operatorname{Spec} \mathbf{C}\left[\frac{y}{x}, \frac{z}{x}\right] \cong \mathbf{A}^2_{\mathbf{C}}, \\ U_y &= \operatorname{Spec} \mathbf{C}\left[\frac{x}{y}, \frac{z}{y}\right] \cong \mathbf{A}^2_{\mathbf{C}}, \text{ and} \\ U_z &= \operatorname{Spec} \mathbf{C}\left[\frac{x}{z}, \frac{y}{z}\right] \cong \mathbf{A}^2_{\mathbf{C}}. \end{aligned}$$

In U_x, $(0, 1)$ corresponds to $(y/x, z/x - 1) \subseteq \mathbf{C}[y/x, z/x - 1]$, and in projective space, it corresponds to $[1 : 0 : 1]$ which corresponds to $(y, z - x)$ a homogeneous ideal in $\mathbf{C}[x, y, z]$. Hence, there are closed points whose corresponding homogeneous ideals are not maximal.

1.2 Review of Algebraic Geometry: Sheaf Cohomology

Remark 1.2.1. Fix a scheme X. We will assume that X is noetherian and separated. Recall that X is separated if the diagonal map $X \xrightarrow{\Delta} X \times X$ is closed - an analog of Hausdorff. The purpose of requiring schemes to be separated is that it guarantees that if $U, W \subseteq X$ are affine open subscemes, then $U \cap W$ is also affine.

Definition 1.2.2 ($\mathcal{O}_X$-module)**.** A sheaf $\mathcal{F}$ on X is called an **$\mathcal{O}_X$-module** provided for each $U \subseteq X$ open, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module.

Example 1.2.3. ${\mathcal{O}_X}^{\oplus d}$ is a free sheaf of $\mathcal{O}_X$-modules.

Example 1.2.4. Another example of an $\mathcal{O}_X$-module is $\mathcal{I} \subseteq \mathcal{O}_X$ an ideal sheaf (which, recall, is a sheaf such that $\mathcal{I}(U) \subseteq \mathcal{O}_X(U)$ is an ideal for all $U \subseteq X$).

Remark 1.2.5. If $X = \operatorname{Spec} R$, every R-module M defines a sheaf of $\mathcal{O}_X$-modules, $\widetilde{M}$, determined by $\widetilde{M}(U_f) = M_f$. Furthermore, note that $\operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \cong \operatorname{Hom}_R(M, N)$.

Remark 1.2.6. Let X be a scheme. Set $\mathbf{Sh}_X$ the category of sheaves on X. Recall from **Definition 1.1.13** that given a map $\varphi : X \to Y$, there is a functor $\varphi_* : \mathbf{Sh}_X \to \mathbf{Sh}_Y$, the pushforward of sheaves. We have a functor in the other direction as well which is a left adjoint to the pushforward functor.

Definition 1.2.7 (inverse image functor)**.** Let $\varphi : X \to Y$ be a morphism of schemes. There is a functor $\varphi^{-1} : \mathbf{Sh}_Y \to \mathbf{Sh}_X$ defined by

$$\varphi^{-1}\mathcal{G}(W) = \varinjlim_{\varphi(W)\subseteq V \text{ open}} \mathcal{G}(V)$$

for $W \subseteq X$ open, $V \subseteq Y$ open, and $\mathcal{G} \in \mathrm{obj}(\mathbf{Sh}_X)$.

☡ **Warning! 1.2.8.** If $\mathcal{G}$ is an $\mathcal{O}_Y$-module, it is not clear that $\varphi^{-1}\mathcal{G}$ is an $\mathcal{O}_X$-module.

Definition 1.2.9 (quasi-coherent)**.** For a scheme $X = (|X|, \mathcal{O}_X)$, call an $\mathcal{O}_X$-module $\mathcal{F}$ **quasi-coherent** provided there is an affine open cover $\{U_i \cong \operatorname{Spec} R_i\}$ such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$. Write $\mathbf{qCoh}_X$ for the category of quasi-coherent $\mathcal{O}_X$-modules.

Definition 1.2.10 (pullback functor)**.** Define the **pullback of sheaves** $\varphi^* : \mathbf{qCoh}_Y \to \mathbf{qCoh}_X$ to be $\varphi^*\mathcal{G} = \varphi^{-1}\mathcal{G} \otimes_{\varphi^{-1}\mathcal{O}_Y} \mathcal{O}_X$ for $\mathcal{G}$ a quasi-coherent sheaf on Y.

Remark 1.2.11. If $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$ for rings R and S, then a map $\varphi : X \to Y$ arises as a map $\varphi^\sharp : S \to R$. By putting φ^* in module language, we have

$$\begin{array}{ccc} R\text{-}\mathbf{mod} & & S\text{-}\mathbf{mod} \\ \| & & \| \\ \mathbf{qCoh}_X & \underset{\varphi^*}{\overset{\varphi_*}{\rightleftarrows}} & \mathbf{qCoh}_Y \end{array}$$

so that $\varphi_* : R\text{-}\mathbf{mod} \to S\text{-}\mathbf{mod}$ is restriction of scalars and $\varphi^* : S\text{-}\mathbf{mod} \to R\text{-}\mathbf{mod}$ is base change.

Remark 1.2.12. In addition to the adjoint pair in **Remark 1.2.6**, we also have another adjunction

$$\operatorname{Hom}_{\mathcal{O}_X}(\varphi^*\mathcal{G}, \mathcal{F}) \cong \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{G}, \varphi_*\mathcal{F})$$

for $\varphi : X \to Y$, $\mathcal{F}$ a sheaf on X, and $\mathcal{G}$ a sheaf on Y.

Theorem 1.2.13 (Grothendieck)**.** *The category* $\mathbf{qCoh}_X$ *is abelian and has enough injectives.*

Remark 1.2.14. Theorem 1.2.13 is exactly what we need to get the standard homological machine off the ground.

Definition 1.2.15 (sheaf cohomology)**.** Let $\mathcal{F}$ be a sheaf on X. Define

$$H^i(X, \mathcal{F}) = \mathbf{R}^i\Gamma(X, \mathcal{F}) = h^i(\Gamma(X, \mathcal{I}^\bullet))$$

for an injective resolution $0 \to \mathcal{F} \to \mathcal{I}^\bullet$.

Example 1.2.16. The following is an application of sheaf cohomology. Fix a scheme $X = (|X|, \mathcal{O}_X)$. Consider an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$ with ideal $I \subseteq \mathcal{O}_X(X)$. Set $Z = V(I) \subseteq X$ the subscheme defined by I. One has a natural short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} \to 0.$$

Denote by $\iota : Z \to X$ the inclusion. One can identify $\iota_*\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}$. We get the associated long exact sequence

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^0(X, \mathcal{O}_X) & \longrightarrow & H^0(X, \iota_*\mathcal{O}_Z) & \xrightarrow{\delta} & H^1(X, \mathcal{I}) & \longrightarrow & \cdots \\ & & \| & & \| & & & & \\ & & \Gamma(X, \mathcal{O}_X) & \xrightarrow{\text{res}} & \Gamma(X, \iota_*\mathcal{O}_Z) & & & & \\ & & & & \| & & & & \\ & & & & \Gamma(Z, \mathcal{O}_Z) & & & & \end{array}$$

So the restriction map is surjective if and only if $\operatorname{im}\delta = 0$. Thus, $H^1(X, \mathcal{I}) = 0$ implies all global sections of Z lift to X.

Remark 1.2.17. We're most interested in quasi-coherent sheaves $\mathcal{F}$ for which $H^i(X, \mathcal{F}) = 0$ for $i > 0$.

Remark 1.2.18. Let's consider $X = \operatorname{Proj} S \cong \mathbf{P}^d_{\mathbf{C}}$ for $S = \mathbf{C}[x_0, ..., x_d]$. Recall that for $U \subseteq X$,

$$\Gamma(U, \mathcal{O}_X) = \left\{ s \mid \text{locally } s = \frac{f}{g} \in \operatorname{Frac} S \text{ with } g(p) \neq 0 \text{ for } p \in U \text{ and } \deg f = \deg g \right\}.$$

Under the identification of X with $\mathbf{P}^d_{\mathbf{C}}$, f/g is a well-defined function if and only if $\deg f = \deg g$.

For each $n \in \mathbf{Z}$, we get a different sheaf $\mathcal{O}_X(n)$ defined by

$$\Gamma(U, \mathcal{O}_X(n)) = \left\{ s \mid \text{locally } s = \frac{f}{g} \text{ with } g(p) \neq 0 \text{ for } p \in U \text{ and } \deg f = \deg g + n \right\}.$$

These are called the Serre twists of $\mathcal{O}_X$.

Example 1.2.19. $\Gamma(X, \mathcal{O}_X(1))$ has a $\mathbf{C}$-basis of $\{x_0, x_1, ..., x_d\}$.

Example 1.2.20. $\Gamma(X, \mathcal{O}_X(2))$ has a $\mathbf{C}$-basis of $\{x_i x_j \mid i, j \in \{0, ..., d\}\}$.

Example 1.2.21. In general, one can see that $\dim_{\mathbf{C}} \Gamma(X, \mathcal{O}_X(n)) = \binom{n+d}{d}$ for $n \geq 0$.

Example 1.2.22. If $n < 0$, $\Gamma(X, \mathcal{O}_X(n)) = \emptyset$, but $1/x_i \in \Gamma(U_{x_i}, \mathcal{O}_X(-1))$.

Theorem 1.2.23 (Serre)**.**

$$H^q\left(\mathbf{P}^d_{\mathbf{C}}, \mathcal{O}_{\mathbf{P}^d_{\mathbf{C}}}(n)\right) = \begin{cases} [\mathbf{C}[x_0, ..., x_d]]_n & \text{if } q = 0; \\ \left[\frac{1}{x_0 \cdots x_d} \mathbf{C}\left[\frac{1}{x_0}, ..., \frac{1}{x_d}\right]\right]_n & \text{if } q = d; \\ 0 & \text{otherwise.} \end{cases}$$

1.3 Review of Algebraic Geometry: Divisors and Line Bundles

Remark 1.3.1. For us, the sheaves $\mathcal{O}_X(n)$ are in a nice class called line bundles. We'll always work in the normal setting; i.e., any scheme X is assumed to be noetherian, separated, irreducible, and normal. Recall that normal schemes satisfy two properties:

1. X is (R_1); i.e., regular in codimension 1. In other words, $\operatorname{codim} \operatorname{Sing} X \geq 2$. For instance, if X is one-dimensional, then it must be smooth; if X is two-dimensional, then it must have isolated singularities; if X is three-dimensional, then it must have no more than a curve of singularities; etc.
2. X is (S_2); i.e., all local rings of X that have codimension at least 2 have depth at least 2.

Definition 1.3.2 (dual module)**.** Let R be a normal domain and M an R-module. Define $M^\vee = \operatorname{Hom}_R(M, R)$, the **dual** of M.

Definition 1.3.3 (reflexive module)**.** There is a natural map $M \to M^{\vee\vee}$. Call M **reflextive** if $M \xrightarrow{\sim} M^{\vee\vee}$.

Example 1.3.4. If M is free, $M^\vee$ is free, and M is reflexive.

Example 1.3.5. For any M, $M^\vee$ is torsion free. Indeed, if $r\varphi = 0$ for $r \neq 0$ and $\varphi \in M^\vee$, then $r\varphi(m) = 0$ for all $m \in M$. As R is a domain and $r \neq 0$, $\varphi(m) = 0$ for all $m \in M$; i.e., $\varphi = 0$.

Consequently, if M is torsion, then $M^\vee = 0$.

Remark 1.3.6. For any M, $M^{\vee\vee}$ is reflexive.

Definition 1.3.7 (sheafy Hom)**.** For two sheaves $\mathcal{F}$ and $\mathcal{G}$, define the **sheafy Hom** $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ to be the sheaf defined by

$$U \mapsto \operatorname{Hom}\left(\mathcal{F}|_U, \mathcal{G}|_U\right).$$

Definition 1.3.8 (dual sheaf)**.** We can define $\mathcal{F}^\vee = \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$.

Definition 1.3.9 (reflexive sheaf)**.** Call $\mathcal{F}$ **reflexive** if the natural map $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism.

Remark 1.3.10. Recall that $\mathcal{F} \cong \widetilde{M}$ locally means that $\mathcal{F}$ is quasi-coherent (**Definition 1.2.9**). $\mathcal{F}$ is coherent if each M is finitely generated.

Theorem 1.3.11 (Hartogs' Principle)**.** *Let X be a normal, irreducible, noetherian scheme and $\mathcal{F}$ a coherent, reflexive $\mathcal{O}_X$-module. For any closed subscheme $Z \subseteq X$, set $U = X \setminus Z$. Denote $\iota : U \hookrightarrow X$. If the codimension of Z is at least 2, then $\mathcal{F} \to \iota_* \left.\mathcal{F}\right|_U$ is an isomorphism.*

Remark 1.3.12. The slogan for **Theorem 1.3.11** [**Hartogs' Principle**] is that reflexive sheaves are determined on codimension 2 sets. For instance, if X is a surface, Z is a collection of discrete points, and $U = X \setminus Z$, then any reflexive $\mathcal{F}$ is determined by $\left.\mathcal{F}\right|_U$.

Remark 1.3.13. This foreshadows the following result: if X is normal, then there is an open set $X_{sm} \subseteq X$ with X_{sm} smooth/regular and of codimension at least 2. That is, any reflexive sheaf on X is determined by X_{sm}.

Remark 1.3.14. Now, to motivate divisors, first consider a curve, X. Given a rational function $s = f/g$ with f and g having no common factors, we can define

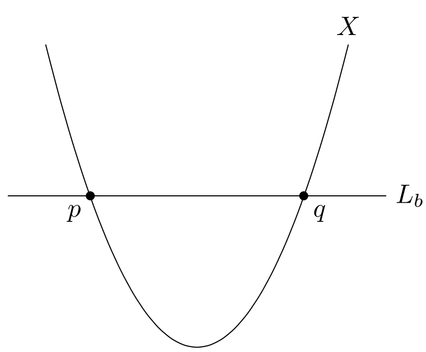
$$\text{Zeros}(s) = \{p \in X \mid f(p) = 0\}$$

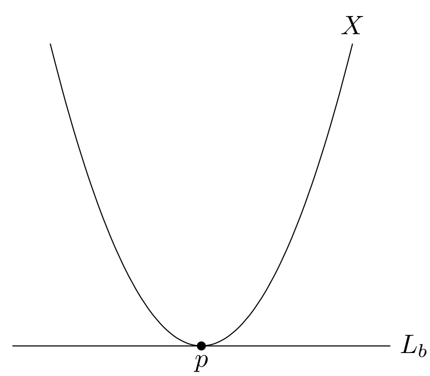
and

$$\text{Poles}(s) = \{p \in X \mid g(p) = 0\}.$$

We know that we should count multiplicities as well.

Example 1.3.15. Let $X = V(y - x^2) \subseteq \mathbf{C}^2$. For each $b \in \mathbf{C}$, we get $L_b = V(y - b)$ and an intersection $X \cap L_b$.

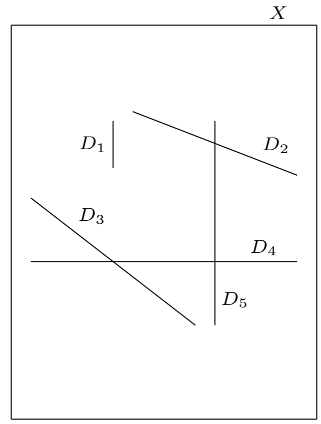




We can build a rational function s with $\text{Zeros}(s) = \{p, q\}$ or $\text{Zeros}(s)$ "equal to the set" $\{p, p\}$.

Definition 1.3.16 (Weil divisor)**.** Fix a scheme X. A **Weil divisor** is a finite formal sum $\sum a_i D_i$ where $a_i \in \mathbf{Z}$ and D_i is an irreducible reduced subscheme of codimension 1.

Example 1.3.17. The following is the cartoon to have in mind:



A Weil divisor could be something like $2D_1 - 3D_2 + D_3 - D_4 + 6D_5$.

Definition 1.3.18 (prime divisor)**.** A single irreducible reduced subscheme of codimension 1, D, is called a **prime divisor**.

Example 1.3.19. Recalling **Example 1.3.15**, for $X = V(y - x^2)$ and $L_b = V(y - b)$ in $\mathbf{C}^2$, $X \cap L_b$ has associated divisors $p + q$ or $2p$. Hence, we have the aforementioned way to track zeros of a rational function s with multiplicity.

Definition 1.3.20 (effective divisor)**.** A divisor $D = \sum a_i D_i$ is said to be **effective** provided $a_i \geq 0$. We write $D \geq 0$ without concern.

Remark 1.3.21. We'll suppress the details, but for $X = \operatorname{Spec} R$, each prime divisor D is defined by a height 1 prime $\mathfrak{q}_D$. If we denote η_D the generic point of D, then we can view $\mathfrak{q}_D$ as the maximal ideal of the local ring $\mathcal{O}_{X,\eta_D}$. If R is normal, then $\mathcal{O}_{X,\eta_D}$ is a discrete valuation ring. There is a natural valuation ν_D on R that measures the zeros and poles of rational functions along D. So for each $s = f/g \in \operatorname{Frac} R$, we can write

$$\operatorname{div}(s) = \sum_{D \text{ prime}} \nu_D(s) \cdot D.$$

Example 1.3.22. Let $X = \operatorname{Spec} \mathbf{C}[x] \cong \mathbf{A}^1_{\mathbf{C}}$. Let

$$s = \frac{x(x-i)^2(x+1)^3}{x(x-i)(x+2i)^2}.$$

Thus $\operatorname{div}(s) = V(x - i) + 3V(x + 1) - 2V(x + 2i)$. To see this, note that, for example, in $\mathbf{C}[x]_{(x+1)}$, one has $s \in (x + 1)^3 \setminus (x + 1)^2$.

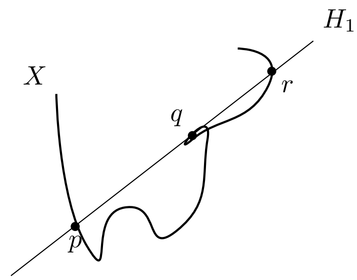
Remark 1.3.23. This perspective globalizes; for a scheme X, set $K(X)$ to be its rational function field (i.e., $K(X) = \mathcal{O}_{X,\eta_X}$. Any $s \in K(X)^\times$ defines a divisor $\operatorname{div}(s)$, and as X is noetherian, $\nu_D(s) \neq 0$ for only finitely many prime divisors.

Definition 1.3.24 (principal divisor)**.** We call such divisors of the form $\operatorname{div}(s)$ for some rational function s **principal**.

Example 1.3.25. Let $X = \mathbf{P}^n_{\mathbf{C}}$ and we work analytically. Here, $K(X)$ is the field of meromorphic functions. Any $\operatorname{div}(s)$ is of the form $\sum a_i D_i + \sum b_j D_j$ for $a_i \geq 0$ and $b_j < 0$. That is, $\sum a_i D_i$ is Zeros(s) and $\sum b_j D_j$ is Poles(s).

Note that for projective space, we see that $\sum a_i + \sum b_j = 0$. This holds algebraically as well; hence, there are lots of divisors which are not principal.

Example 1.3.26. Fix $X \subsetneq \mathbf{P}^n_{\mathbf{C}}$ and assume that X is large; i.e., X is not contained in a hyperplane. Each hyperplane H defines a divisor $D_H(X)$ by considering $X \cap H$. Below is a sample cartoon:



Here, $D_{H_1}(X) = p+2q+r$, and for another hyperplane H_2, $D_{H_2}(X)$ would be another divisor with coefficients also adding to 4, by Bézout's theorem. If we write two hyperplanes as $H_1 = V(f)$ and $H_2 = V(g)$, then $D_{H_1}(X) - D_{H_2}(X) = \operatorname{div}(f/g)$; that is, their difference is principal. Furthermore, note that for $s, s' \in K(X)$, $\operatorname{div}(ss') = \operatorname{div}(s) + \operatorname{div}(s')$. This motivates the following.

Definition 1.3.27 (divisor class group)**.** For X noetherian, normal, and irreducible, denote by $\operatorname{Div}(X)$ the group of Weil divisors. The principal divisors, $\operatorname{Prin} X$, are a subgroup. Thus, define $\operatorname{Cl}(X) = \operatorname{Div}(X)/\operatorname{Prin}(X)$, the **divisor class group**. That is, we say $D \equiv D'$ for Weil divisors D and D' if and only if $D = D' + \operatorname{div}(s)$ for some s, and we say D is linearly equivalent to D'.

Remark 1.3.28. There is always a group homomorphism $\operatorname{Cl}(X) \to \mathbf{Z}$, the degree map, sending a class

$$\left[\sum n_i D_i\right] \mapsto \sum n_i.$$

This is well-defined, since $\deg(\operatorname{div}(s)) = 0$, by **Example 1.3.25**.

Example 1.3.29. $\operatorname{Cl}(\mathbf{P}^n_{\mathbf{C}}) \cong \mathbf{Z}$.

Theorem 1.3.30 (Hartshorne)**.** *A noetherian normal domain R is a UFD if and only if* $\operatorname{Cl}(\operatorname{Spec} R) = 0$.

Remark 1.3.31. Divisors are also linked to the concept of fractional ideals.

Definition 1.3.32 (fractional ideal)**.** For a domain R, a **fractional ideal** is a finitely generated submodule of $\operatorname{Frac} R$.

Example 1.3.33. $(1/2)\mathbf{Z} \subseteq \mathbf{Q}$ is a finitely generated submodule of $\mathbf{Q} = \operatorname{Frac} \mathbf{Z}$.

Remark 1.3.34. Each Weil divisor D defines a sheaf in the following way. Let D be a Weil divisor; we get a sheaf $\mathcal{O}_X(D)$ where $\Gamma(U, \mathcal{O}_X(D)) = \{s \in K(X) \mid \operatorname{div} s|_U + D|_U \geq 0\}$.

Example 1.3.35. Let X be a curve. Let $D = 2p + q - 3r$ for points $p, q, r \in X$. If $s = f/g \in K(X)$, then we have $\operatorname{div} s = \sum a_i D_i$. If $s \in \Gamma(X, \mathcal{O}_X(D))$, then we need $\operatorname{div}(s) + D$ to be effective; i.e., $\operatorname{div} s + D \geq 0$ on X. This puts constraints on a_p, a_q, and a_r, while all other a_i just need to be nonnegative. See that $a_p + 2 \geq 0$, $a_q + 1 \geq 0$, and $a_r - 3 \geq 0$. Thus, $a_p \geq -2$, $a_q \geq -1$, and $a_r \geq 3$. Thus, s is allowed poles at p and q of order at most 2 and 1, but must have a zero of order at least 3 at r.

Remark 1.3.36. If D is effective, then sections of $\mathcal{O}_X(D)$ are allowed poles, and $\mathcal{O}_X(-D)$ must have zeros along D. One can identify $\mathcal{O}_X(-D)$ with an ideal sheaf $\mathcal{O}_X(-D) = \mathfrak{a}_D \subseteq \mathcal{O}_X$ defining D; we call it antieffective.

Example 1.3.37. If $D = D_i$ is prime in $X = \operatorname{Spec} R$, then $\mathfrak{a}_D \subseteq R$ and $V(\mathfrak{a}_D) = D$. For instance, let $R = \mathbf{C}[x, y]$ and $D = V(x - y)$. Then $\mathfrak{a}_D = (x - y)$, and we write $\mathfrak{a}_D = R(-D)$.

Remark 1.3.38. We'll see next that $\mathcal{O}_X(D)$ is a fractional ideal. To see this, we'll use duals. Fix $X = \operatorname{Spec} R$ and $M \subseteq \operatorname{Frac} R$ a submodule.

Definition 1.3.39 (colon)**.** Define $R :_{\operatorname{Frac} R} M = \{f/g \in \operatorname{Frac} R \mid (f/g)M \subseteq R\}$.

Lemma 1.3.40. *If M is torsion free and rank 1, then* $M^\vee = \operatorname{Hom}_R(M, R) \cong R :_{\operatorname{Frac} R} M$.

Proof. As $\operatorname{rank} M = 1$, $\operatorname{Frac} R \otimes M \cong \operatorname{Frac} R$. As M is torsion free, the map $M \to M \otimes \operatorname{Frac} R$ is injective. To identify $\operatorname{Hom}_R(M, R)$ with $R :_{\operatorname{Frac} R} M$, note that any $s \in R :_{\operatorname{Frac} R} M$ defines a multiplication map $M \to R$ via $m \mapsto sm$; i.e., s yields an element of $\operatorname{Hom}_R(M, R)$.

Conversely, view

$$\begin{aligned}\operatorname{Hom}_R(M, R) &\cong \operatorname{Hom}_R(M, R) \otimes R \\ &\subseteq \operatorname{Hom}_R(M, R) \otimes \operatorname{Frac} R \\ &\cong \operatorname{Hom}_{\operatorname{Frac} R}(M \otimes \operatorname{Frac} R, R \otimes \operatorname{Frac} R) \\ &\cong \operatorname{Hom}_{\operatorname{Frac} R}(\operatorname{Frac} R, \operatorname{Frac} R) \\ &\cong \operatorname{Frac} R,\end{aligned}$$

with the final isomorphism given via multiplication. So any element of $\operatorname{Hom}_R(M, R)$ is given by $s \in \operatorname{Frac} R$, and by definition such $s \in R :_{\operatorname{Frac} R} M$. ☐

Remark 1.3.41. So if $X = \operatorname{Spec} R$ and we define $R(D) = \Gamma(X, \mathcal{O}_X(D))$ for a Weil divisor D, then, recalling that each prime divisor has an associated height 1 prime ideal by **Remark 1.3.21**, we see that if D is prime, then $R(D) \cong D^\vee \cong R :_{\operatorname{Frac} R} D$. Essentially, $\operatorname{div} s + D \geq 0$ if and only if $sD \subseteq R$.

Example 1.3.42. Let $R = \mathbf{Z}$, so $\operatorname{Frac} R = \mathbf{Q}$, and let $D = p\mathbf{Z}$. As a result, we have that

$$D^\vee = \operatorname{Hom}(p\mathbf{Z}, \mathbf{Z}) \cong \mathbf{Z} :_{\mathbf{Q}} p\mathbf{Z} \cong \frac{1}{p}\mathbf{Z},$$

a fractional ideal.

Lemma 1.3.43. *Let $X = \operatorname{Spec} R$, let D be a Weil divisor, and write $R(D) = \Gamma(X, \mathcal{O}_X(D))$. The sheaf $R(D)$ is reflexive, and $\mathcal{O}_X(D)$ is a reflexive $\mathcal{O}_X$-module.*

Remark 1.3.44. We also state the following facts:

- If D is a prime Weil divisor on $X = \operatorname{Spec} R$, then $R(D) \cong D^\vee \cong R :_{\operatorname{Frac} R} D$. Here, we conflate D with its associated height 1 prime ideal.
- $R(-D) \cong D$.
- One can check that for divisors A and B, $R(A + B) \cong (R(A) \cdot R(B))^{\vee\vee}$. Recall that $M^{\vee\vee}$ is the reflexive closure of M. In particular, if $D = \sum a_i D_i \geq 0$ is effective, then $R(-D) = (\prod {D_i}^{a_i})^{\vee\vee}$ and $R(D) = R(-D)^\vee$ is a fractional ideal.
- $D = \operatorname{div} s$ is principal if and only if $R(D) \cong (1/s)R$ is free.

Remark 1.3.45. One can define $\operatorname{Cl}(R)$ as the multiplicative group of all height 1 prime fractional ideals modulo principal fractional ideals. From here, it's clear that R is a UFD if and only if $\operatorname{Cl}(R) = 0$.

Example 1.3.46. Let $R = k[x^2, xy, y^2] \cong k[a, b, c]/(ac - b^2)$ for k a field. We have a divisor in the scheme $X = \operatorname{Spec} R$ given by $D = (x^2, xy)$ which is not locally principal, but notice that $R(-2D) = (D^2)^{\vee\vee}$ and $D^2 = (x^4, x^2y^2, x^3y) = (x^2 \cdot x^2, x^2 \cdot y^2, x^2 \cdot xy) = x^2R$, and $R(2D) = (1/x^2)R \cong (x^2R)^\vee$.

Theorem 1.3.47. *Two divisors D_1 and D_2 are linearly equivalent if and only if $R(D_1) \cong R(D_2)$. All the more, this holds for all normal, noetherian, irreducible, separated schemes.*

Proof sketch. We prove one direction. If $D_1 = D_2 + \operatorname{div} s$ for some s - that is, D_1 and D_2 are linearly equivalent - then we claim that $R(D_1)s \cong R(D_2)$. Given $f \in R(D_1)$, we have $\operatorname{div} f + D_1 \geq 0$, so $\operatorname{div}(fs) + D_1 = \operatorname{div} f + \operatorname{div} s + D_1 \geq \operatorname{div} s$. Thus, $\operatorname{div}(fs) + D_1 - \operatorname{div} s = \operatorname{div}(fs) + D_2 \geq 0$, so $fs \in R(D_2)$. On the other hand, let $g \in R(D_2)$; then $\operatorname{div} g + D_2 \geq 0$, so $\operatorname{div}(g/s) + D_1 = \operatorname{div} g + D_1 - \operatorname{div} s = \operatorname{div} g + D_2 \geq 0$, so $g/s \in R(D_1)$, and therefore $g \in R(D_1)s$. Hence the double inclusion is shown.

We omit the other direction. ☐

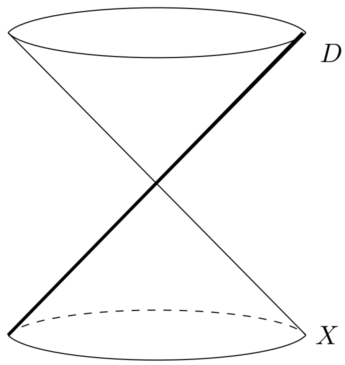
Remark 1.3.48. The main takeaway is that divisors are a linear way to understand the geometry of X.

Definition 1.3.49 (Cartier divisor)**.** A Weil divisor D on a scheme X is called **Cartier** provided that there is an open cover $\{U_i\}$ of X such that $D|_{U_i} = \operatorname{div} f_i$ is principal.

Example 1.3.50. For the most basic example, any principal divisor is Cartier.

Example 1.3.51. For a nonexample, consider the scheme $X = \operatorname{Spec} k[x^2, xy, y^2]$ with divisor $D = (x^2, xy)$ from **Example 1.3.46**. D is not locally principal, so it is not Cartier.

Example 1.3.52. For another divisor that is not Cartier, let $X = V(x^2 + y^2 - z^2) \subseteq \mathbf{A}^3$ be the cone. Form the divisor D by intersection with $V(x - z)$. D is not Cartier as it is not principal near the cone point. A function at the cone point needs to vanish in y to order 2.



Lemma 1.3.53. *Let $X = \operatorname{Spec} R$. D is Cartier if and only if $R(D)$ is projective.*

Proof. Locally, D is principal if and only if $R(D)$ is free of rank 1. □

Definition 1.3.54 (trivialization)**.** We call the data $\{U_i, f_i\}$ of a Cartier divisor D a **trivialization** of D.

Remark 1.3.55. Note that given a trivialization $\{U_i, f_i\}$, on $U_i \cap U_j$, we have $\operatorname{div} f_i|_{U_i \cap U_j} = \operatorname{div} f_j|_{U_i \cap U_j}$; i.e., f_i/f_j is regular.

Remark 1.3.56. For a Weil divisor D, we claim that we have the following bijection:

$$\begin{aligned} \left\{s \mid s \in H^0(X, \mathcal{O}_X(D))\right\} &\leftrightarrow \{D' \mid D' \geq 0, D' \equiv D\} \\ s &\mapsto D + \operatorname{div} s. \end{aligned}$$

To see this claim, set K_X the sheaf of locally constant functions with values in $K(X)$. Set K_X^* the sheaf of nonzero section in K_X. View $\mathcal{O}_X^*$ as the sheaf of locally constant sections of K_X without poles. Thus $\mathcal{O}_X^* \subseteq K_X^*$. Hence, we have a short exact sequence

$$0 \to \mathcal{O}_X^* \to K_X^* \to K_X^*/\mathcal{O}_X^* \to 0.$$

We now claim that a global section of $H^0(X, K_X^*/\mathcal{O}_X^*)$ defines a Cartier divisor. Indeed, as $K_X^*/\mathcal{O}_X^*$ is only defined via sheafification, a global section arises from an open cover $\{U_i\}$ and $f_i \in K_X^*(U_i)$ such that gluing occurs; i.e., $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ in $K_X^*(U_i)/\mathcal{O}_X^*(U_i)$. That is, f_i/f_j is regular on $U_i \cap U_j$. Therefore, $\{U_i, f_i\}$ is indeed a trivialization of a Cartier divisor.

Now, this divisor is principal if and only if the associated global section is in

$$\operatorname{im}\left(H^0\left(X, K_X^*\right) \to H^0\left(X, K_X^*/\mathcal{O}_X^*\right)\right).$$

As K_X^* is flasque, $H^1(X, K_X^*) = 0$; i.e., $H^0(X, K_X^*) \to H^0(X, K_X^*/\mathcal{O}_X^*) \to H^1(X, \mathcal{O}_X^*) \to 0$ is exact. That is, we have an identification

$$H^1(X, \mathcal{O}_X^*) \leftrightarrow \{D \mid D \text{ is a Cartier divisor}\}/\sim,$$

where $\sim$ denotes linear equivalence.

Definition 1.3.57 (Picard group)**.** The group $H^1(X, \mathcal{O}_X^*) \cong \{D \mid D \text{ is Cartier}\}/\sim$ is called the **Picard group** of X. Write $\operatorname{Pic}(X)$.

Theorem 1.3.58. *If X is smooth, then all Weil divisors are Cartier.*

Proof. Without loss of generality, it suffices to consider prime divisors D. Set $\mathfrak{a}_D$ to be the ideal sheaf defining D. Recall that D is principal if and only if $\mathfrak{a}_D$ is principal. At each point $x \in X$, pass to the local ring $\mathcal{O}_{X,x}$. Set $\mathfrak{a}_x = \mathfrak{a}_D \mathcal{O}_{X,x}$. As X is smooth, $\mathcal{O}_{X,x}$ is regular and $\mathfrak{a}_x$ has height 1. By Krull Hauptidealsatz, $\mathfrak{a}_x$ is principal. Now, the sheaf $\mathfrak{a}_D$ has stalk $\mathfrak{a}_x$ which is principal; recall that

$$\mathfrak{a}_x = \varinjlim_{x \in U} \mathfrak{a}_D(U).$$

As X is noetherian, $\mathfrak{a}_x$ is finitely generated, so we can pick U_x sufficiently small so that $\mathfrak{a}_x \cong \Gamma(U_x, \mathfrak{a}_D)$; but then $D|_{U_x}$ is principal, so $D|_{U_x} = \operatorname{div} f_x$. Now cover X by $\{U_i, f_i\}$ which is a trivialization. □

Remark 1.3.59. In **Theorem 1.3.58**, the hypothesis that X is smooth is not needed in certain circumstances. If $\mathcal{O}_{X,x}$ is a UFD, then all height 1 ideals are principal, and the proof holds.

Definition 1.3.60 (factoral)**.** We call X **factoral** if $\mathcal{O}_{X,x}$ is a UFD for all $x \in X$.

Corollary 1.3.61. *If X is factoral, then all Weil divisors are Cartier.*

Remark 1.3.62. We now turn to the second topic of the section, line bundles, and see how Cartier divisors are related to line bundles.

Definition 1.3.63 (locally free sheaf)**.** A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is **locally free** if there is an open cover $\{U_i\}$ such that $\mathcal{F}|_{U_i}$ is free; i.e., $\mathcal{F}|_{U_i} \cong {\mathcal{O}_X}^{\oplus d_i}$.

Definition 1.3.64 (rank of a locally free sheaf)**.** If $\mathcal{F}$ is a locally free sheaf, $\mathcal{F}|_{U_i} \cong {\mathcal{O}_X}^{\oplus d}$, and d is independent, then we say $\mathcal{F}$ is **locally free of rank** d.

Definition 1.3.65 (vector bundle)**.** If $\mathcal{F}$ is a locally free sheaf of rank d, then we say $\mathcal{F}$ is a **vector bundle**.

Definition 1.3.66 (line bundle)**.** If $\mathcal{F}$ is a vector bundle and the rank of $\mathcal{F}$ is 1, then $\mathcal{F}$ is called a **line bundle**.

Remark 1.3.67. In topology, a vector bundle is a space E with a surjective map $\pi : E \to X$ such that each fiber $\pi^{-1}(x)$ is a vector space for all $x \in X$. A section is a map $s : U \to E$, where $U \subseteq X$ is open and $\pi(s(x)) = x$ for all $x \in U$. To each topological vector bundle, we can associate its sheaf of sections $\mathcal{E}(U) = \{s : U \to X \mid s \text{ is a section}\}$. Observe that if X is a scheme, then $\mathcal{E}(U)$ is locally free of rank $\dim \pi^{-1}(x)$ for all $x \in X$. Hence be warned when definitions diverge.

Example 1.3.68. On any X, $\mathcal{O}_X$ is a line bundle.

Example 1.3.69. If $X = \mathbf{P}^d_{\mathbf{C}}$, then each Serre twist $\mathcal{O}_X(n)$ is a line bundle.

Remark 1.3.70. We will see how, essentially, line bundles are the same as Cartier divisors. Each Cartier divisor D has a trivialization $\{U_i, f_i\}$ and a sheaf $\mathcal{O}_X(D)$ with

$$\Gamma(U, \mathcal{O}_X(D)) = \{s \in K(X) \mid \operatorname{div} s|_U + D|_U \geq 0\},$$

and $\mathcal{O}_X(D)|_{U_i} \cong \mathcal{O}_X(\operatorname{div} f_i)|_{U_i} \cong \mathcal{O}_X|_{U_i}$, since $\operatorname{div} f_i$ is principal of rank 1. And so, $\mathcal{O}_X(D)$ is a line bundle.

However, if $D \equiv D'$, then $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$. So $\mathcal{O}_X(D)$, a line bundle, does not determine D, only its class in $\operatorname{Pic} X$. Therefore, we'd like a way, given a line bundle $\mathcal{L}$, to produce D such that $\mathcal{L} \cong \mathcal{O}_X(D)$. As $\mathcal{L}$ is locally free and rank 1, pick an open cover $\{U_i\}$ and generators $\mathcal{L}|_{U_i} \cong g_i \mathcal{O}_{U_i}$.

Note that D given by $\{U_i, g_i\}$ is a Cartier divisor with $\mathcal{O}_X(D) \cong \mathcal{L}$, but the choice of g_is is arbitrary.

Now, any section $s \in \Gamma(X, \mathcal{L})$ defines $s|_{U_i} = f_i g_i$ in $\Gamma(U_i, \mathcal{O}_{U_i})$, and $\{U_i, f_i\}$ also defines a Cartier divisor D with $\mathcal{O}_X(D) \cong \mathcal{L}$. Note that $f_i g_i \in \mathcal{O}_{U_i}$; i.e., $\operatorname{div} f_i g_i \geq 0$, so each D is effective. Thus, there is a bijection

$$\begin{aligned} \Gamma(X, \mathcal{L}) &\leftrightarrow \{D \mid D \text{ is a Cartier divisor}, D \geq 0, \mathcal{O}_X(D) \cong \mathcal{L}\} \\ s &\mapsto \{U_i, f_i\}. \end{aligned}$$

As Cartier divisors form the group $\operatorname{Pic} X$, we expect a group operation on line bundles.

Lemma 1.3.71. *Let $\mathcal{E}$ be a vector bundle, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Set $\mathcal{E}^\vee = \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$. One has $\mathcal{H}om(\mathcal{E}, \mathcal{F}) \cong \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{F}$.*

If $\mathcal{L}$ is a line bundle, one has $\mathcal{L} \otimes \mathcal{L}^\vee \cong \mathcal{H}om(\mathcal{L}, \mathcal{L}) \cong \mathcal{O}_X$.

Proof. If suffices to work on stalks, or equivalently, on small open sets U. We can further restrict U so that $\mathcal{E}(U)$ is free. Here, it suffices to prove that $\mathrm{Hom}(\mathcal{E}(U), \mathcal{F}(U)) \cong \mathrm{Hom}(\mathcal{E}(U), \mathcal{O}_X(U)) \otimes \mathcal{F}(U)$. Set $R = \mathcal{O}_X(U)$ (R for ring), $F = \mathcal{E}(U)$ (F for free), and $M = \mathcal{F}(U)$ (M for module). We thus see that

$$\begin{aligned} \mathrm{Hom}(F, R) \otimes M &\longrightarrow \mathrm{Hom}(F, M) \\ \varphi \quad \otimes \quad m &\longmapsto \psi : F \longrightarrow M \\ &\qquad\quad x \longmapsto \varphi(x) \cdot m \end{aligned}$$

is an isomorphism, because F is free. $\square$

Remark 1.3.72. By **Lemma 1.3.71**, the set of line bundles $\mathcal{L}$ forms a group under $\otimes$ with inverses $(-)^\vee = \mathcal{H}om(-, \mathcal{O}_X)$. This group is isomorphic to $\mathrm{Pic}\, X$, since

1. if $\mathcal{L} \cong \mathcal{O}_X(D)$ and $\mathcal{L}' \cong \mathcal{O}_X(D')$, then $\mathcal{L} \otimes \mathcal{L}' \cong \mathcal{O}_X(D + D')$, and
2. if $\mathcal{L} \cong \mathcal{O}_X(D)$, then $\mathcal{L}^\vee \cong \mathcal{O}_X(-D)$.

Therefore, the moral is that the group operation on line bundles is thought of multiplicatively, while on divisors is thought of additively.

Example 1.3.73. Let $X = \mathbf{P}^d_{\mathbf{C}}$. On X, all Weil divisors are Cartier, and the degree map, which is $\mathrm{Cl}(X) = \mathrm{Pic}(X) \to \mathbf{Z}$ defined by $\sum n_i D_i \mapsto \sum n_i$ has an inverse map $m \mapsto \mathcal{O}_X(m) \cong \mathcal{O}_X(mH)$ for any hyperplane H in X. By picking coordinates, we can assume $H = V(x_0)$. It is a direct calculation that $\mathcal{O}_X(H) \cong \mathcal{O}_X(1)$.

Remark 1.3.74. The motivation is that line bundles provide nice embeddings. If we fix a line bundle $\mathcal{L}$ on X and suppose $s_0, ..., s_d \in \Gamma(X, \mathcal{L})$ are sections such that at each point $x \in X$, $s_i(x) \neq 0$ for some i (given such a condition, we say $\mathcal{L}$ is globally generated, or that $\mathcal{L}$ is basepoint free), then we get a morphism

$$\begin{aligned} \Phi : X &\to \mathbf{P}^d_{\mathbf{C}} \\ x &\mapsto [s_0(x) : s_1(x) : \cdots : s_d(x)]. \end{aligned}$$

If furthermore, the sections s_i separate tangents (we say $\mathcal{L}$ is very ample), then Φ is a closed immersion which identifies X as a subscheme of $\mathbf{P}^d_{\mathbf{C}}$. We call $\Phi^*(\mathcal{O}_{\mathbf{P}^d_{\mathbf{C}}}(1)) = \mathcal{O}_X(1)$.

Conjecture 1.3.75 (Fujita)**.** *If X is a smooth, irreducible scheme, $\mathcal{L}$ has a power which is very ample (we say $\mathcal{L}$ is ample), and $\omega_X \cong \mathcal{O}_X(K_X)$ where K_X is a "canonical divisor," then*

1. *$\omega_X \otimes \mathcal{L}^{\dim X + 1}$ is basepoint free, and*
2. *$\omega_X \otimes \mathcal{L}^{\dim X + 2}$ is very ample.*

The sheaf $\omega_X \otimes \mathcal{L}^n$ is called an adjoint bundle.